Thermostatistic properties of a *q*-generalized Bose system trapped in an *n*-dimensional harmonic oscillator potential

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The thermostatistic properties of a q-generalized boson system trapped in an n-dimensional harmonic oscillator potential are studied, based on the generalized statistic distribution derived from Tsallis' entropy. The density of states, total number of particles, critical temperature at which Bose-Einstein condensation occurs, internal energy, and heat capacity at constant volume are derived. The characteristics of Bose-Einstein condensation of the system are discussed in detail. It is found from the results obtained here that the thermostatistic properties of such a system depend closely on parameter q, dimensional number of the space, and frequency of the harmonic oscillator; and the external potential has a great influence on the thermostatistic properties of the system.

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I. INTRODUCTION

The q-generalized statistical mechanics proposed by Tsallis [1] and developed by many researchers [2–14] has become a powerful tool to deal with some systems which are more complex than a standard ideal gas and present long-range interactions and/or long-duration memory. In recent years, it has been successfully used to study the thermostatistic properties of many physical systems such as self-gravitating stellar systems [15,16] with q lower than $\frac{7}{9}$, low-dimensional dissipative systems [17] with $q<1$, the Lévy flight random diffusion [18,19] with $\frac{5}{3}<q<3$, the galaxy model of the generalized Freeman disk [20] with $q=-1$, the electron plasma two-dimensional turbulence [15] with $q=\frac{1}{2}$, the cosmic background radiation [21] and correlated themes [22], the linear response theory [23], the solar neutrinos [24], thermalization of electron-phonon systems [25], etc. A large number of significant results obtained have shown that parameter q can play an important role in such studies.

On the other hand, it is well known that in a many body system of bosons, it is possible to have an off-diagonal long-range order (ODLRO) of the reduced density matrices in coordinate space representation [26]. The onset of ODLRO points at quantum phase and quantum phase transitions in a many body system. The well-known example for a many body bosonic system is Bose-Einstein condensation (BEC) [27], which was predicted by Bose and Einstein more than 70 years ago. Owing to the development of techniques to trap and cool atoms, BEC of the Bose gas was experimentally demonstrated in magnetic traps of rubidium [28,29], lithium [30], and sodium [31] gases in 1995. This has caused a sensation throughout the physics community. Many authors have studied the thermostatistic properties of the boson systems from both theory and experiment and a great number of important results have been obtained. In the research, the external potential plays an important role. Especially in experimental settings, trapping potentials are well approximated by the potential of a harmonic oscillator [32] and so it has been used to investigate many interesting problems of BEC [33–42], such as spectral equivalence of bosons and fermions [34], Hartree-Fock calculations of BEC of ^{7}Li atoms [36], density of states for BEC [41], etc.

In past years, the q-generalized statistical mechanics has been used to investigate the generalized BEC of a q-boson system and some important results have been obtained [43–45]. However, to our knowledge, these studies are mainly restricted to a free q-boson system. Investigations have seldom been done on the properties of the trapped q bosons, which may be more closely related to the sensational experiments of BEC in ultracold trapped Bose gases [28–31]. Thus, it is significant to investigate the thermostatistic properties of the trapped q-boson system.

In the present paper, the generalized Bose-Einstein (BE) distribution function derived from Tsallis' entropy will be used to study the thermostatistic properties of a q-boson system trapped in an n-dimensional harmonic oscillator potential. The paper is organized as follows. In Sec. II, the total number of particles and critical temperature of a q-boson system are derived analytically. The characteristics of the critical temperature are analyzed in detail and the conditions under which BEC may occur are determined. In Sec. III, the total energy and heat capacity of the system are calculated. The characteristics of the heat capacity for different temperature regions are discussed. Some important results are obtained. In Sec. IV, the properties of an ordinary boson system trapped in an n-dimensional harmonic oscillator potential are

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directly deduced. Finally, some important conclusions are summed up.

II. TOTAL NUMBER OF PARTICLES AND CRITICAL TEMPERATURE

It is well known that the q-generalized statistics characterized by a parameter q rely on the so-called Tsallis' entropy [1]

$$S_q = k\frac{1-\Sigma_{i=1}^{W} p_i^q}{q-1}, \tag{1}$$

where $q\in R$, k is the Boltzmann constant, p_i is the probability of a particle at the ith state, and W is the total number of states of the system. Within the approximation method called factorization approach, the generalized BE distribution can be written in an important formalism [46–48]:

$$n_q = \frac{1}{[1+(q-1)\beta(\varepsilon-\mu)]^{1/(q-1)}-1}, \tag{2}$$

where n_q is the average occupation number at a state with energy, ε, $\beta=1/(kT)$, T is the absolute temperature, and μ is the chemical potential. When $q=1$, Eq. (2) becomes the well-known BE distribution.

Now let us consider a q-generalized Bose system trapped in an n-dimensional harmonic oscillator potential. The Hamiltonian of a single particle in the system can be written as

$$H = ap^s + \frac{1}{2}m\sum_{i=1}^{n}|\omega_i r_i|^2, \tag{3}$$

where a and s are two positive constants to describe the kinetic energy of the system. For example, $a=1/(2m)$, $s=2$, and $a=c$, $s=1$ correspond to the nonrelativistic and ultrarelativistic cases, respectively, c is the speed of light, m is the particle mass, and ω_i is the frequency of harmonic oscillators.

For an n-dimensional harmonic oscillator potential with spherical symmetric, the Hamiltonian of a single particle in the system may be simplified as

$$H = ap^s + \tfrac{1}{2}m\omega^2|r|^2. \tag{4}$$

It is worthwhile to point out that such a consideration does not lose generality. In fact, it can be proven easily that if one is interested in recovering the case of $\omega_i\neq\omega_j$, it is sufficient to replace ω^n with $\Pi_{i=1}^{n}\omega_i$ in the following equations.

When the number of particles in the system is large and the potential energy of particles in a trap is much smaller than their kinetic energy (this condition is often satisfied), the Thomas-Fermi semiclassical approximation is valid [49]. Thus, the sums over quantum states may be replaced by integrals over phase space. The total number of quantum states for $H\leqslant\varepsilon$ may be expressed as

$$\Sigma(\varepsilon) = \frac{g}{h^n}\int_{H\leqslant\varepsilon}\prod_{i=1}^{n}(dr_i dp_i) = \frac{g\pi^n}{(h\omega)^n\Gamma(n/2+1)a^{n/s}(m/2)^{n/2}}\frac{\Gamma(n/s+1)}{\Gamma(\lambda+1)}\varepsilon^{\lambda}, \tag{5}$$

where

$$\lambda = n/s + n/2. \tag{6}$$

The derivative of Eq. (5) with respect to ε yields the expression of the density of states for an ideal system trapped in an n-dimensional harmonic oscillator potential, i.e.,

$$D(\varepsilon) = \frac{g\pi^n}{(h\omega)^n\Gamma(n/2+1)a^{n/s}(m/2)^{n/2}}\frac{\Gamma(n/s+1)}{\Gamma(\lambda)}\varepsilon^{\lambda-1} \tag{7}$$

and g is the degree of degeneracy. By using Eq. (7), the total number of particles of the system can be written as

$$N = N_0 + \int n_q D(\varepsilon)d\varepsilon = N_0 + \frac{g}{x_s^n}\frac{\pi^{n/2}}{(m/2)^{n/2}}g_{q,\lambda}(z_q), \tag{8}$$

where N_0 is the number of particles in the ground state ($\varepsilon=0$), $z_q=[1+(1-q)\beta\mu]^{1/(1-q)}$ is called the generalized fugacity [45], $x_s=[\Gamma(n/2+1)/\Gamma(n/s+1)]^{1/n}[a(h\omega)^s/\pi^{s/2}]^{1/s}\beta^{\lambda/n}$ is called the generalized thermal wavelength [50] which is independent of q, and

$$g_{q,\lambda}(z_q) = \begin{cases} \displaystyle\sum_{j=1}^{\infty}\frac{z_q^{j+(1-q)\lambda}}{(1-q)^{\lambda}}\frac{\Gamma(j/(1-q)+1)}{\Gamma(j/(1-q)+\lambda+1)} & (q<1) \\ \displaystyle\sum_{j=1}^{\infty}\frac{z_q^{j-(1-q)\lambda}}{(q-1)^{\lambda}}\frac{\Gamma(j/(q-1)-\lambda)}{\Gamma(j/(q-1))} & (q>1) \end{cases} \tag{9}$$

is called the q-generalized Bose integral. In order to guarantee the q-generalized Bose integral to be larger than zero, $q<(\lambda+1)/\lambda$ must be satisfied. It can be seen from Eq. (9) that $g_{q,\lambda}(z_q)$ is a monotonically increasing function of z_q and the maximum value of z_q is equal to 1 when $\mu=0$.

When temperature is not lower than the critical temperature of BEC, most of the particles in the system are at the excited states and the number of particles in the ground state is macroscopically negligible, so we obtain

$$N \approx N_e = \frac{g}{x_s^n}\frac{\pi^{n/2}}{(m/2)^{n/2}}g_{q,\lambda}(z_q) = \frac{g}{x_{s,c}^n}\frac{\pi^{n/2}}{(m/2)^{n/2}}\xi_q(\lambda), \tag{10}$$

where $x_{s,c}=[\Gamma(n/2+1)/\Gamma(n/s+1)]^{1/n}[a(h\omega)^s/\pi^{s/2}]^{1/s}(1/kT_{q,c})^{\lambda/n}$ is the generalized thermal wavelength at the critical temperature $T_{q,c}$ of BEC, which can be defined as the same as an ordinary boson system where $q=1$ and is given by

$$T_{q,c}=\frac{1}{k}\left[\frac{N}{g}\frac{(m/2)^{n/2}a^{n/s}(h\omega)^n}{\pi^n}\frac{\Gamma(n/2+1)}{\Gamma(n/s+1)}\frac{1}{\xi_q(\lambda)}\right]^{1/\lambda}, \tag{11}$$

and

$$\xi_q(\lambda)=\begin{cases}\sum_{j=1}^{\infty}\frac{1}{(1-q)^\lambda}\frac{\Gamma(j/(1-q)+1)}{\Gamma(j/(1-q)+\lambda+1)} & (q<1)\\ \sum_{j=1}^{\infty}\frac{1}{(q-1)^\lambda}\frac{\Gamma(j/(q-1)-\lambda)}{\Gamma(j/(q-1))} & (q>1)\end{cases} \tag{12}$$

is the generalized Riemann zeta function.

When $q=1$, Eq. (12) reduces to the Riemann zeta function $\xi(\lambda)$, i.e.,

$$\lim_{q\to 1}\xi_q(\lambda)=\xi(\lambda) \tag{13}$$

and Eq. (11) can be simplified as

$$T_{1,c}=\frac{1}{k}\left[\frac{N}{g}\frac{(m/2)^{n/2}a^{n/s}(h\omega)^n}{\pi^n}\frac{\Gamma(n/2+1)}{\Gamma(n/s+1)}\frac{1}{\xi(\lambda)}\right]^{1/\lambda}\equiv T_c, \tag{14}$$

which is just the critical temperature of an ordinary boson system trapped in an n-dimensional harmonic oscillator potential. When $n=3$ and $s=2$, important results obtained in Refs. [51,52], $T_c=(h\omega/2\pi k)[N/\xi(3)]^{1/3}$, can be directly derived from Eq. (14).

By using Eq. (14), Eq. (11) can be expressed as a simple relation:

$$\frac{T_{q,c}}{T_c}=\left[\frac{\xi(\lambda)}{\xi_q(\lambda)}\right]^{1/\lambda}. \tag{15}$$

This result is similar to that in Ref. [45], but the parameters related to the external potential are included in parameter λ. It has been proved [45] that $\xi_q(\lambda)$ is convergent when $\lambda>1$. This implies the fact that if and only if $\lambda>1$, BEC of a q-boson system trapped in a harmonic oscillator potential can occur. It can be clearly seen from Eq. (6) that within a harmonic oscillator potential trap, BEC of a q-boson system can occur more easily than that of a q-boson system without any external potential. For example, for the nonrelativity case $(s=2)$, only when the dimension of space is larger than 2 can BEC of a free q-boson system occur. However, with the help of a harmonic oscillator potential trap, BEC of a q-boson system with $n=2$ can occur.

Using Eq. (15), we can plot the curves of $T_{q,c}/T_c$ with respect to parameter q, as shown in Fig. 1. It can be seen from Fig. 1 that $T_{q,c}/T_c$ decreases with the increase of q. This means that the smaller the parameter q is, the more easily the BEC of a system can occur. When $q<1$, $\xi_q(\lambda)<\xi(\lambda)$, $T_{q,c}>T_c$, and $T_{q,c}/T_c$ increases with the increase of λ. This implies the fact that BEC of a q-boson system will occur more easily than that of an ordinary boson system and the larger the parameter λ, the more easily the BEC of a q-boson system will occur. It is clearly seen from Eq. (6) that

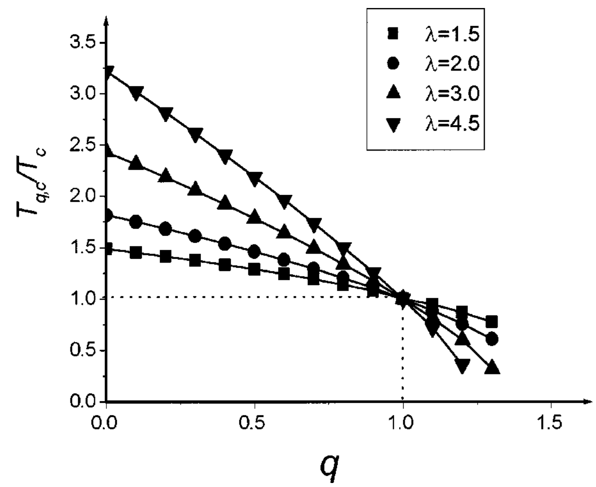

FIG. 1. The curves of the critical temperature varying with parameter q for some different values of λ.

parameter λ is proportional to the dimensional number of space, so that the larger the dimensional number of space of a q-boson system is, the more easily the BEC of a q-boson system will occur. The results obtained here will be helpful in searching some q-boson systems which have a higher critical temperature of BEC. When $q>1$, the results are just the contrary. In addition, from the fact that $T_{q,c}/T_c$ decreases with the increase of q, it can be estimated that the correlation between the particles is repulsive for $q>1$ and attractive for $q<1$. This important conclusion is the same as that obtained in Ref. [44].

When $T<T_{q,c}$, one part of the particles condenses in the ground state. From Eqs. (8) and (10), one can obtain the particle number at the excited states,

$$N_e=N\left(\frac{T}{T_{q,c}}\right)^\lambda, \tag{16}$$

and the fraction of condensation in the ground state,

$$\frac{N_0}{N}=1-\left(\frac{T}{T_{q,c}}\right)^\lambda. \tag{17}$$

These results are similar to those of an ordinary boson system, but the critical temperatures for the q-boson and ordinary boson systems are different.

III. INTERNAL ENERGY AND HEAT CAPACITY

We now continue to derive the total energy of the system. By using Eq. (5), the total energy of the system is given by

$$\begin{aligned}E&=\int n_q\varepsilon D(\varepsilon)d\varepsilon\\ &=\frac{g\pi^n}{(h\omega)^n\Gamma(n/2+1)a^{n/s}(m/2)^{n/2}}\frac{\Gamma(n/s+1)}{\Gamma(\lambda)}\\ &\quad\times\int\frac{\varepsilon^\lambda d\varepsilon}{[1+(q-1)\beta(\varepsilon-\mu)]^{1/(q-1)}-1}\end{aligned}$$

$$=\frac{g\pi^n\Gamma(n/s+1)}{(h\omega)^n\Gamma(n/2+1)a^{n/s}(m/2)^{n/2}}\frac{\lambda}{\beta^{\lambda+1}}g_{q,\lambda+1}(z_q). \tag{18}$$

Comparing Eqs. (18) and (8), one can easily find that when $T\leqslant T_{q,c}$,

$$E=\lambda kTN\left(\frac{T}{T_{q,c}}\right)^{\lambda}\frac{\xi_q(\lambda+1)}{\xi_q(\lambda)}; \tag{19}$$

when $T>T_{q,c}$,

$$E=\lambda kTN\frac{g_{q,\lambda+1}(z_q)}{g_{q,\lambda}(z_q)}. \tag{20}$$

Substituting Eqs. (19) and (20) into the expression of the heat capacity at constant volume

$$C_v=\left[\frac{\partial E}{\partial T}\right]_{N,V}, \tag{21}$$

one can find that the heat capacity at constant volume is given by

$$C_v(T\leqslant T_{q,c})=\left(\frac{\partial E(T\leqslant T_{q,c})}{\partial T}\right)_{N,V}=\lambda(\lambda+1)k\;\;N\left(\frac{T}{T_{q,c}}\right)^{\lambda}\frac{\xi_q(\lambda+1)}{\xi_q(\lambda)}, \tag{22}$$

when $T\leqslant T_{q,c}$;

$$C_v(T>T_{q,c})=\left(\frac{\partial E(T>T_{q,c})}{\partial T}\right)_{N,V}=\lambda(\lambda+1)\;\;Nk\frac{g_{q,\lambda+1}(z_q)}{g_{q,\lambda}(z_q)}-\lambda^2Nk\frac{g_{q,\lambda}(z_q)}{g_{q,\lambda-1}(z_q)}, \tag{23}$$

when $T>T_{q,c}$ (a detailed derivation is given in the Appendix).

It is seen from Eqs. (22) and (23) that the heat capacity at constant volume of the q-boson system trapped in a harmonic oscillator potential is dependent not only on the temperature but also on parameters λ and q. For given parameters λ and q, one can generate the curves of C_v varying with $T/T_{c,q}$, as shown in Fig. 2. It is important to note that some important results can be deduced from the curves in Fig. 2.

(i) The heat capacity at constant volume C_v of the q-boson system trapped in a harmonic oscillator potential at all temperatures increases with the increase of parameters λ and q. It shows that C_v of a q-boson system for $q<1$ is smaller than that of an ordinary boson system at all temperatures, because the correlation between the particles in such a q-boson system is attractive. When $q>1$, the result is just the contrary.

(ii) In the region of $T<T_{q,c}$, the heat capacity at constant volume C_v is a monotonically increasing function of temperature, which is similar to the case of an ordinary boson system.

(iii) In the region of $T>T_{q,c}$, the heat capacity at constant volume C_v is very different from the case of an ordinary boson system. When $q<1$, the heat capacity at constant volume C_v is a monotonically decreasing function of temperature. When $q>1$, the heat capacity at constant volume C_v is not a monotonic function of temperature. It first decreases and then increases with T, so there is a minimal value for C_v. This result has been observed experimentally [44]. When $T\gg T_{q,c}$, the heat capacity at constant volume C_v of the q-boson system trapped in a harmonic oscillator potential is not equal to $(n/s)k$ and is still dependent on parameters λ and q, while the heat capacity at constant volume C_v of an ordinary boson system is always equal to $(n/s)k$. This shows once again that q is an important parameter which describes implicitly the interaction of the particles in the system. When $q\neq 1$, the correlation between the particles in the system must be considered even at high temperatures.

(iv) When $T=T_{q,c}$, we obtain

$$\left[\frac{\Delta C_V}{Nk}\right]_{T=T_{q,c}}=\lambda^2\frac{\xi_q(\lambda)}{\xi_q(\lambda-1)} \tag{24}$$

from Eqs. (22) and (23). Equation (24) shows clearly that when $\lambda\leqslant 2$, the heat capacity at the critical temperature is continuous; when $\lambda>2$, the heat capacity at the critical temperature is discontinuous and the jump at the critical temperature is dependent on parameters λ and q. This point can be clearly seen from Fig. 2.

IV. A SPECIAL CASE

When $q=1$, as expected, one can directly derive the thermodynamic properties of an ordinary boson system trapped in an n-dimensional harmonic oscillator potential. For example, from Eqs. (16), (17), and (19)–(21), one can obtain the ground state fraction

$$\frac{N_0}{N}=1-\left(\frac{T}{T_c}\right)^n, \tag{25}$$

the internal energy

$$E=\begin{cases}\lambda ktN\left(\frac{T}{T_c}\right)^{\lambda}\frac{\xi_{\lambda+1}(z)}{\xi_\lambda(z)} & (T\leqslant T_c)\\[2mm] \lambda ktN\frac{g_{\lambda+1}(z)}{g_\lambda(z)} & (T>T_c),\end{cases} \tag{26}$$

the heat capacity at constant volume

$$C_v=\begin{cases}\lambda(\lambda+1)kN\left(\frac{T}{T_c}\right)^{\lambda}\frac{\xi_{\lambda+1}(z)}{\xi_\lambda(z)} & (T\leqslant T_c)\\[2mm] \lambda(\lambda+1)kN\frac{g_{\lambda+1}(z)}{g_\lambda(z)}-\lambda^2kN\frac{g_\lambda(z)}{g_{\lambda-1}(z)} & (T>T_c),\end{cases} \tag{27}$$

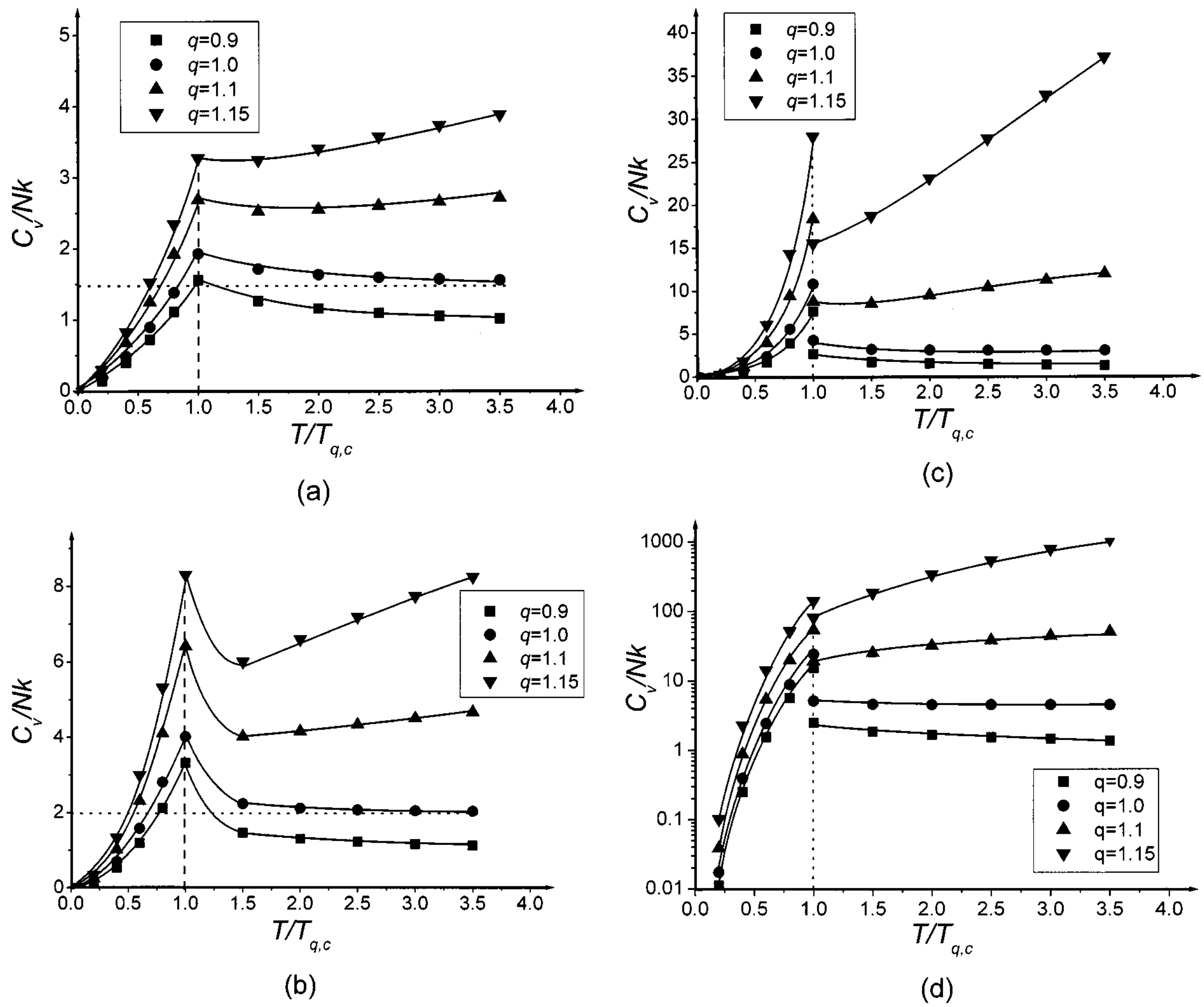

FIG. 2. The curves of the heat capacity varying with $T/T_{q,c}$ for some different values of q. (a) $\lambda=1.5$, (b) $\lambda=2$, (c) $\lambda=3$, and (d) $\lambda=4.5$.

and the jump at the critical temperature

$$\left[\frac{\Delta C_v}{Nk}\right]_{T=T_{q,c}}=\lambda^2\frac{\xi(\lambda)}{\xi(\lambda-1)}, \tag{28}$$

where $g_n(z)=\Sigma_{j=1}^{\infty}z^j/j^n$ is the Bose integral [53]. If one does not consider any external potential, the results in Ref. [54] can be derived from the above results.

V. CONCLUSIONS

With the help of the q-generalized BE distribution function and the density of states of an ideal Bose system trapped in an n-dimensional harmonic oscillator potential, we have successfully derived the analytic expressions for the thermodynamic parameters of a q-generalized boson system trapped in an n-dimensional harmonic oscillator potential by introducing some significant physical parameters such as the generalized Bose integral, generalized Riemann zeta function, generalized wavelength, and so on. The important parameters include the total number of particles, critical temperature, fraction of condensation in the ground state, internal energy, heat capacity at constant volume, the jump of the heat capacity at the critical temperature, and so on.

It is significant to find the following important conclusions from the analytic expressions obtained in this paper.

(i) The thermostatistic properties of a q-generalized boson system trapped in an n-dimensional harmonic oscillator potential are closely dependent not only on parameters q and n/s but also on the external potential.

(ii) Only when $\lambda>1$ can BEC of a q-generalized boson system occur.

(iii) The smaller the parameter q, the higher the critical temperature of BEC for a q-generalized boson system.

(iv) When $1<\lambda\leqslant 2$, the heat capacity at the critical temperature of BEC is continuous; while for $2<\lambda$, the heat capacity at the critical temperature of BEC is discontinuous and there is a jump, which is dependent on parameters λ and q.

(v) In the region of $T\gg T_{q,c}$, the heat capacity at constant volume of a q-generalized boson system $(q\neq 1)$ is not equal to $(n/s)k$. This is very different from the case of an ordinary boson system $(q=1)$.

The results obtained here are very general. They are suitable not only for a q-generalized boson but also for an ordinary boson system trapped in any-dimensional harmonic oscillator potential.

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APPENDIX

Using the relation $(\partial N/\partial T)_V=0$ and Eq. (10), one can obtain

$$\partial g_{q,\lambda}(z_q)/\partial T=\frac{\partial g_{q,\lambda}(z_q)}{\partial z_q}\frac{\partial z_q}{\partial T}=-\frac{\lambda}{T}g_{q,\lambda}(z_q). \quad (A1)$$

From Eq. (9), we can derive

$$\frac{\partial g_{q,\lambda}(z_q)}{\partial z_q}=\frac{1}{z_q^q}g_{q,\lambda-1}(z_q). \quad (A2)$$

Solving Eqs. (A1) and (A2) gives

$$\frac{\partial z_q}{\partial T}=-z_q^q\frac{\lambda}{T}\frac{g_{q,\lambda}(z_q)}{g_{q,\lambda-1}(z_q)}. \quad (A3)$$

Similarly, one can obtain

$$\partial g_{q,\lambda+1}(z_q)/\partial T=\frac{1}{z_q^q}g_{q,\lambda}(z_q)\frac{\partial z_q}{\partial T}$$
$$=-\lambda\frac{g_{q,\lambda}(z_q)}{T}\frac{g_{q,\lambda}(z_q)}{g_{q,\lambda-1}(z_q)}. \quad (A4)$$

Using Eqs. (20) and (21), one can derive

$$C_v(T>T_{q,c})=\left(\frac{\partial E(T>T_{q,c})}{\partial T}\right)_{N,V}=\lambda Nk\frac{g_{q,\lambda+1}(z_q)}{g_{q,\lambda}(z_q)}+\lambda NkT\frac{[\partial g_{q,\lambda+1}(z_q)/\partial T]g_{q,\lambda}(z_q)-[\partial g_{q,\lambda}(z_q)/\partial T]g_{q,\lambda+1}(z_q)}{[g_{q,\lambda}(z_q)]^2}. \quad (A5)$$

Substituting Eqs. (A3) and (A4) into Eq. (A5) yields Eq. (23).

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